dependence for $r/R_e \geqq 10$ (Fig. 4). An additional criterion is suggested by the plume length beyond which the particle and gas dynamics become uncoupled. This is essentially the condition after the particles pass into free molecule flow. On the plume axis, and for the largest particles, free molecule flow prevails for distances beyond about 7 exit radii downstream for the several plumes examined.

Since the present model deals only with the mass flux of particles, it contains no information about particle sizes or particle velocities. When this distribution is valid the particles have reached what is essentially a limiting velocity, which is less than the gas limiting velocity and is different for different particle sizes. The mass flux distribution is therefore directly applicable only to contamination estimates because heating and pressure estimates require an additional specification of velocity.

Finally, it is recognized that the particular form chosen for the fractional mass flux is not unique and there may be others which might improve the comparison. For example, the error function profile does not satisfy the conditions $w(1) = 1$ and $h(1) = 0$ identically. A function which satisfies these conditions and gives a reasonable fit to the numerical solutions is $w(\xi) = \sin^2(\xi\pi/2)$. However, this choice did not yield as satisfactory a fit to $h(\xi)$ for $\xi > 0.5$ as that given by Eq. (7).

The most important conclusion of this work is that the particle plume is accurately described as a spherical source flow. The particular form of the mass flux distribution given here is based on a set of calculations for conical nozzles and may not be uniformly satisfactory for other motors. However, the calculations against which these results were checked are representative of motors which are in common use which suggests that the simple particle plume mass flux model proposed here will be adequate for a majority of the cases of interest.

References

[1] Kliegel, J. R. and Nickerson, G. R., "Axisymmetric Two-Phase Perfect Gas Performance Program," Rept. 2874-6006-R000, April 1967, TRW Systems, Redondo Beach, Calif.

[2] Hoffman, R. J. et al., "Plume Contamination Effects Prediction," AFRPL-TR-71-109, Dec. 1971, McDonnell Douglas Astronautics Co., Huntington Beach, Calif.

[3] Hill, J. A. F. and Draper, J. S., "Analytical Approximation for the Flow from a Nozzle into a Vacuum," *Journal of Spacecraft and Rockets*, Vol. 3, No. 10, Oct. 1966, pp. 1552–1554.

[4] Pearce, B. E. and Wang, H. E., "Utilization of Solid Rocket Motors for Space Shuttle, Vol. VII, Staging Motor Exhaust Plume Effects on Shuttle Vehicle," Rept. ATR-73(7313-02)-1, Vol. VII, Aug. 1972, The Aerospace Corp., El Segundo, Calif.

Spin Stability of Torque-Free Systems Containing Rotors

T. R. Kane*
Stanford University, Stanford, Calif.

THE method described recently[1] for generating stability criteria applicable to spinning motions of certain torque-free systems cannot be used directly for systems containing free or driven rotors, for the development of the method involves the requirements that there be neither cyclic coordinates nor time-dependent constraints. Thus, dual-spin vehicles, satellites containing gyroscopic nutation dampers, etc., are excluded from consideration in Ref. 1. It is the purpose of this Note to indicate how such "gyroscopic" systems can be brought under purview of an extended form of the theory set forth in Ref. 1.

Received September 28, 1973.
Index category: Spacecraft Attitude Dynamics and Control.
* Professor of Applied Mechanics.

All symbols not defined in what follows have the same meaning as in Ref. 1. A development closely analogous to that described in Ref. 1, and based on the principles discussed in Ref. 2, can be used to establish the validity of the results to be stated.

The system $\bar{S}$ to be examined consists of the system S considered in Ref. 1 together with N axisymmetric rotors R_ρ $(\rho = 1,\ldots,N)$. The rotors are attached to S in such a way that the mass center and axis of rotation of R_ρ remain fixed in reference from A, and R_ρ rotates with constant angular speed σ_ρ relative to A, whenever $\bar{S}$ performs a quasi-gyrostatic motion, that is, a motion during which the internal configuration of S remains unaltered. Such a motion is called a simple spin if ω has a constant magnitude and is at all times parallel to a principal axis of inertia of $\bar{S}$ for the mass center $\bar{S}^*$ of $\bar{S}$.

To formulate stability conditions for a motion of simple spin of $\bar{S}$, one may proceed as follows:

a) Select coordinates $q_1,\ldots,q_n$ governing the internal configuration of S in such a way that $q_1 = \cdots = q_n = 0$ during the simple spin to be investigated, and form V and D_r $(r = 1,\ldots,n)$ such that the requirements imposed by Eqs. (2–4) of Ref. 1 are satisfied.

b) Let Z_1, Z_2, Z_3 be a set of mutually perpendicular axes intersecting at $\bar{S}^*$, parallel to principal axes of inertia of $\bar{S}$ for $\bar{S}^*$ when $q_1 = \cdots = q_n = 0$, and numbered such that Z_1 plays the role of spin axis; let $\mathbf{z}_i$ be a unit vector pointing in the positive direction of Z_i $(i = 1,2,3)$; and let Z be a reference frame in which Z_1, Z_2, Z_3 are fixed.

c) Form I_{jk} (omit I_{23}) and $\mathbf{h}$ in accordance with

$$I_{jk} = \mathbf{z}_j \cdot \mathbf{I} \cdot \mathbf{z}_k \cdot (j,k = 1,2,3)$$

$$\mathbf{h} = \sum_{\rho=1}^{n} \sigma_\rho J_\rho \mathbf{u}_\rho$$

where $\mathbf{I}$ is the inertia dyadic of $\bar{S}$ for $\bar{S}^*$, J_ρ is the axial moment of inertia of R_ρ, and $\mathbf{u}_\rho$ is a unit vector parallel to the symmetry axis of R_ρ and having the same direction as the angular velocity of R_ρ relative to Z whenever $\bar{S}$ is performing a quasi-gyrostatic motion. Evaluate $\tilde{I}_{jj}$, $\tilde{I}_{1j,r}$, $\tilde{I}_{11,rs}$, $\tilde{V}_{,r}$, $\tilde{V}_{,rs}$, $\tilde{\mathbf{h}}$, $\tilde{\mathbf{h}}_{,r}$, and $\tilde{\mathbf{h}}_{,rs}$ $(j = 1, 2, 3; r, s = 1,\ldots,n)$, where the tilde denotes evaluation at $q_1 = \cdots = q_n = 0$ and a subscript comma followed by one or more letters signifies partial differentiation with respect to internal variables, these differentiations being performed in reference frame Z in the case of $\mathbf{h}$.

d) Form $a_{1j,r}$, $a_{11,rs}$, $\tilde{h}_j$, $\tilde{h}_{j,r}$, $\tilde{h}_{1,rs}$, and $\tilde{I}_{,rs}$ in accordance with

$$a_{11,r} = 0, \quad a_{1j,r} = \tilde{I}_{1j,r}/(\tilde{I}_{11} - \tilde{I}_{jj}) \quad (j = 2,3)$$

$$a_{11,rs} = a_{12,r}a_{12,s} + a_{13,r}a_{13,s} \quad (r,s = 1,\ldots,n)$$

$$\tilde{h}_j = \mathbf{h} \cdot \mathbf{z}_j, \quad \tilde{h}_{j,r} = \tilde{\mathbf{h}}_{,r} \cdot \mathbf{z}_j - \tilde{h}_1 a_{ij,r} \quad (j = 1,2,3; r = 1,\ldots,n)$$

$$\tilde{h}_{1,rs} = \tilde{\mathbf{h}}_{,rs} \cdot \mathbf{z}_1 + \tilde{\mathbf{h}}_{,r} \cdot (a_{12,s}\mathbf{z}_2 + a_{13,s}\mathbf{z}_3) + \tilde{h}_1 a_{11,rs} + \tilde{\mathbf{h}}_{,s} \cdot (a_{12,r}\mathbf{z}_2 + a_{13,r}\mathbf{z}_3) \quad (r,s = 1,\ldots,n)$$

e) Verify that the simple spin under consideration can, in fact, occur, by ascertaining that, in accordance with "external" equations of motion

$$\tilde{h}_2 = \tilde{h}_3 = 0$$

whereas to satisfy "internal" equations of motion

$$\tilde{V}_{,r} - 2^{-1}(H - \tilde{h}_1)^2 \tilde{I}_{11,r}/\tilde{I}_{11}^{\,2} - (H - \tilde{h}_1)\tilde{h}_{1,r}/\tilde{I}_{11} = 0 \quad (r = 1,\ldots,n)$$

where H is the magnitude of the angular momentum of $\bar{S}$ with respect to $\bar{S}^*$ in N.

f) Form α_2, α_3, β_{2r}, β_{3r}, and γ_{rs} in accordance with

$$\alpha_2 = H(\tilde{h}_1 - H)/\tilde{I}_{11} + H^2/\tilde{I}_{33}, \quad \alpha_3 = H(\tilde{h}_1 - H)/\tilde{I}_{11} + H^2/\tilde{I}_{22}$$

$$\beta_{2r} = -H\tilde{h}_{3,r}/\tilde{I}_{33}, \quad \beta_{3r} = H\tilde{h}_{2,r}/\tilde{I}_{22} \quad (r = 1,\ldots,n)$$

$$\gamma_{rs} = \tilde{V}_{,rs} + \tilde{h}_{1,rr}\tilde{h}_{1,s}/\tilde{I}_{11} + \tilde{h}_{2,r}\tilde{h}_{2,s}/\tilde{I}_{22} + \tilde{h}_{3,r}\tilde{h}_{3,s}/\tilde{I}_{33} + (H - \tilde{h}_1)[(\tilde{I}_{11,r}\tilde{h}_{1,s} + \tilde{I}_{11,s}\tilde{h}_{1,r})/\tilde{I}_{11} - \tilde{h}_{1,rs}]/\tilde{I}_{11} - (H - \tilde{h}_1)^2(\tilde{I}_{11}^{\,-2}\tilde{I}_{11,rs} - \tilde{I}_{11}^{\,-3}\tilde{I}_{11,r}\tilde{I}_{11,s})/2 \quad (r,s = 1,\ldots,n)$$

g) Formulate stability inequalities by using Sylvester's criteria to insure the positive-definiteness of the matrix δ given by

$$\delta = \begin{bmatrix} \alpha & \beta \\ \beta^T & \gamma \end{bmatrix}$$

where

$$\alpha = \begin{bmatrix} \alpha_2 & 0 \\ 0 & \alpha_3 \end{bmatrix} \qquad \beta = \begin{bmatrix} \beta_{21}\beta_{22}\cdots\beta_{2n} \\ \beta_{31}\beta_{32}\cdots\beta_{3n} \end{bmatrix}$$

$$\gamma = \begin{bmatrix} \gamma_{11}\gamma_{12}\cdots\gamma_{in} \\ \gamma_{21} \\ \vdots \\ \gamma_{n1} \end{bmatrix}$$

h) Formulate "isolation conditions" by determining the circumstances under which the simple spin under consideration is an isolated quasi-gyrostatic motion in the following sense: let 0 be the origin of the $2(n+1)$ dimensional vector space described in Sec. VII of Ref. 1. Then a simple spin is an isolated quasi-gyroscopic motion if there exists a neighborhood of 0 in which 0 is the only point corresponding to a quasi-gyroscopic motion.

The last paragraph of Ref. 1 now describes the conclusions that can be drawn regarding the stability of a simple spin, provided d and e of Ref. 1 be replaced with g, while f of Ref. 1 is replaced with h.

References

[1] Teixeira-Filho, D. R. and Kane, T. R., "Spin Stability of Torque-Free Systems – Part I," *AIAA Journal*, Vol. 11, No. 6, June 1973, pp. 862–867.

[2] Pringle, R., Jr., "Stability of Force-Free Motion of a Dual Spin Spacecraft," *AIAA Journal*, Vol. 7, No. 6, June 1969, pp. 1054–1063.

Derivatives of Eigenvalues and Eigenvectors for a General Matrix

CARL S. RUDISILL*
Clemson University, Clemson, S.C.

I. Introduction

IN the optimal design of systems where the dynamic stability and/or response of the system is a function of several parameters it is often desirable to know the derivatives of the eigenvalues and eigenvectors of a characteristic equation of the system. Rudisill and Bhatia[1,2] utilized the first and second derivatives of the eigenvalues of the equations for steady-state oscillation of an aircraft structure to find the first and second derivatives of the flutter velocity. These derivatives of the flutter velocity were used in gradient search procedures to find the minimum mass structure for a given flutter velocity.

For self-adjoint systems Wittrick[3] derived expressions for the first derivatives of the eigenvalues, and Fox and Kapoor[4] derived expressions for the first derivatives of the eigenvectors. For nonself adjoint-systems Rogers[5] derived expressions for the first derivative of the eigenvalues and the eigenvectors. Rudisill and Bhatia,[1,2] and Plaut and Huseyin[6] derived expressions for the first derivatives of the eigenvalues and eigenvectors and the second derivatives of the eigenvalues. Garg[7] developed a method for finding the first derivatives of the eigenvalues and eigenvectors which requires the solution of $2(n+1)$ equations for an n degree of freedom system.

In the references previously cited the expressions for any one of the derivatives of the eigenvectors of a nonself-adjoint system requires (with the exception of Garg's[7] method) the value of all of the left-hand and right-hand eigenvectors. For large degree-of-freedom systems the expense of finding all of these eigenvectors may be excessive when the derivatives of only one of the eigenvectors are needed. It is the purpose of this Note to derive expressions for the derivatives of the eigenvalues and eigenvectors which are expressions of only one left-hand and one right-hand eigenvector. The method may be extended to find any order of derivative of the eigenvalue and eigenvector.

II. Analysis

Consider the matrix equation

$$(A-\lambda_i B)U_i = 0 \tag{1}$$

where A and B are $n\times n$ matrices and $\lambda_i (i = 1,2,\ldots,n)$ are assumed to be distinct eigenvalues of the characteristic equation

$$\det|A-\lambda_i B| = 0 \tag{2}$$

and U_i is the right-hand eigenvector corresponding to λ_i. The left-hand eigenvector V_i corresponding to λ_i is such that

$$V_i'(A-\lambda_i B) = 0 \tag{3}$$

where the prime denotes the transpose of the matrix.

Taking the partial derivative of Eq. (1) with respect to the jth parameter yields the relation

$$(A_{,j}-\lambda_{i,j}B-\lambda_i B_{,j})U_i+(A-\lambda_i B)U_{i,j} = 0 \tag{4}$$

The first partial derivative of λ_i may now be found by premultiplying Eq. (4) by the transpose of the left-hand eigenvector V_i, then substituting Eq. (3) into the results and finally solving for the partial derivative

$$\lambda_{i,j} = V_i'(A_{,j}-\lambda_i B_{,j})U_i/(V_i' BU_i) \tag{5}$$

The second partial derivative of the eigenvalue λ_i may be found by differentiating Eq. (4) with respect to the kth parameter, premultiplying by the transpose of the left-hand eigenvector V_i, making use of Eq. (3) and then solving for the second partial derivative

$$\begin{aligned}\lambda_{i,jk} = [&V_i'(A_{,jk}-\lambda_{i,j}B_{,k}-\lambda_{i,k}B_{,j}-\lambda_i B_{,jk})U_i+ \\ &V_i'(A_{,j}-\lambda_{i,j}B-\lambda_i B_{,j})U_{i,k}+ \\ &V_i'(A_{,k}-\lambda_{i,k}B-\lambda_i B_{,k})U_{i,j}]/(V_i' BU_i)\end{aligned} \tag{6}$$

Rudisill and Bhatia,[1,2] and Paut and Huseyin[8] developed expressions similar to Eqs. (5) and (6) for the first and second derivatives, they also developed expressions for the derivatives of the eigenvectors which were functions of all n of the eigenvectors U_i and V_i. Next a method will be developed for finding the derivative of U_i without prior knowledge of the other $n-1$ right- and left-hand eigenvectors.

If all of the eigenvalues of Eq. (2) are distinct then there are n distinct linearly independent eigenvectors U_i. The rank of $A-\lambda_i B$ is then $n-1$ and there are only $n-1$ components of U_i which are unique and from Eq. (4) there are only $n-1$ components of $U_{i,j}$ which are unique. Since the length of an eigenvector is arbitrary, the components of the eigenvector may be rendered unique by imposing the constraint

$$U_i'U_i = 1 \tag{7}$$

then

$$U_i'U_{i,j} = 0 \tag{8}$$

Equations (4) and (8) may be combined to form the relations

$$\begin{bmatrix} A-\lambda_i B \\ \hline U_i' \end{bmatrix} U_{i,j} = -\begin{bmatrix} A_{,j}-\lambda_{i,j}B-\lambda_i B_{,j} \\ \hline 0 \end{bmatrix} U_i \tag{9}$$

Received October 23, 1973. This research was supported by NASA Research Grant NGR-41-001-027. The author wishes to express his appreciation to R. V. Doggett Jr. of the NASA Langley Research Center for his advice on the preparation of this Note.

Index categories: Structural Dynamics Analysis; Structural Design, Optimal.

* Associate Professor of Mechanical Engineering. Member AIAA.